THE

MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.; E. T. WHITTAKER, M.A.; W. E. HARTLEY, B.A.

LONDON:

GEORGE BELL & SONS, YORK ST., COVENT GARDEN,

AND BOMBAY.

VOL. II. MAY, 1903. No. 39.

ON THE REPRESENTATION OF IMAGINARY POINTS BY REAL POINTS ON A PLANE.

IN plane coordinate geometry, with a system of rectangular axes, I wish to press the claim of points whose coordinates are given by complex numbers to be represented on the plane in just the same way as all other points in the plane.

It will doubtless be conceded that when we plot a point from its coordinates (x, y) we are in reality plotting its position by means of the complex quantity $x+iy$. This we habitually do when x and y are simple, or 'real' lengths, and my contention is that we ought to extend the privilege to *all* points whatever.

If we allow ourselves to do this when x and y are complex numbers, we can always represent 'imaginary' points by real ones. Such plotted points may be called the representatives of the imaginary points; or, for brevity, they may be considered as being actually the imaginary points, which will thus be real in position, and imaginary only in the sense that they are not visibly situated in the locus whose equation they satisfy by virtue of their particular complex coordinates.

Thus, let $(m+ip, n+iq)$ be the complex coordinates of a point. We plot it as $(m+ip)+i(n+iq)$, *i.e.* $(m-q)+i(n+p)$, its simple coordinates being $(m-q, n+p)$.

The conjugate complex, viz. $(m-ip, n-iq)$ would similarly have the simple coordinates $(m+q, n-p)$.

In this way the pair of complexes $(m\pm ip, n\pm iq)$, and their pair of simple correlatives $(m\mp q, n\pm p)$, both uniquely represent the positions of the two points given by the complex relations $(m+in)\pm i(p+iq)$.

There is a perfect 2 to 2 correspondence between the first and second pair of points.

But there is a distinction.

The points $(m+ip, n+iq)$ and $(m-ip, n-iq)$ satisfy the equation

$$\frac{x-m}{p}=\frac{y-n}{q},$$

while the points $(m-q, n+p)$ and $(m+q, n-p)$ satisfy the equation

$$p(x-m)+q(y-n)=0;$$

i.e. the first pair of points are imaginary points on one line, and the second pair are real points on a line at right angles to the first. Both lines pass

through (m, n). The distance of the first pair of points from (m, n) is $i\sqrt{(p^2+q^2)}$, and the distance of the second pair from (m, n) is $\sqrt{(p^2+q^2)}$.

Hence, two conjugate imaginary points on a line are represented by a pair of real points which are each other's *images* in the line, at a distance from it equal to the modulus of the purely imaginary part of the complex coordinates, their mid-point having as coordinates the purely real parts of the same coordinates.

Now the imaginary points on a line are doubly infinite in number, covering the whole plane. The above investigation shows that they do not lie haphazard on it, but that conjugate points are 'image' points in the line. As a corollary it follows that the plane can be stratified, as it were, into a series of loci consisting of conjugate imaginary points on the line, which will have the line as an axis of symmetry and will cross it at *right angles*, if at all, unless they have some singularity, such as a cusp or double point, where they cross it. And, conversely, any locus which has the line as an axis of symmetry consists of a series of conjugate imaginary points on the line.

Now take the case of a pair of imaginary points on a circle. This we can best do by considering the intersections of a non-cutting straight line with the circle.

Evidently the circle is only one of a coaxial system, cutting the line in the same two points. Hence the limiting points of the system are the representatives of the imaginary points of intersection. Consequently the imaginary points of intersection lie on the radius which cuts the line orthogonally, at distances (r_1, r_2) from the centre given by the equations $r_1r_2=a^2$, and $r_1+r_2=2b$, where a is the radius of the circle, and b the distance of the line from the centre of the circle.

By moving the line in any systematic manner, we can obtain a *locus* of imaginary points which will (except for singularities) cross the circle orthogonally if at all. Thus, if we move the line parallel to itself, the locus is a diameter of the circle perpendicular to the series of moving lines; if we rotate it round any point on it, we obtain an orthogonal circle, and so on. All loci so obtained will have their parts which are outside the circle the inverses of the parts which are inside the circle.

Now take the case of a conic. We will treat this analytically by finding the points where the ellipse

$$b^2x^2+a^2y^2=a^2b^2 \quad \ldots\ldots(1)$$

is cut by any line $bx\cos\phi+ay\sin\phi=kab, \quad \ldots\ldots(2$

where k is greater than 1.

We shall find that if we move the line parallel to itself, the intersection points lie on a confocal conic which cuts the given conic orthogonally at the point of contact of the parallel tangent

$$bx\cos\phi+ay\sin\phi=ab. \quad \ldots\ldots(3)$$

Consequently the whole plane can be stratified, as it were, into a series of confocal conics, each confocal being the locus of the pairs of imaginary points in which a series of parallel external lines cut the given conic.

To prove this, solve (1) and (2) simultaneously, remembering that k is greater than unity.

We find, for the points of intersection,

$$\left.\begin{aligned} x&=ka\cos\phi\pm ia\sin\phi\sqrt{(k^2-1)}\\ y&=kb\sin\phi\mp ib\cos\phi\sqrt{(k^2-1)}\end{aligned}\right\}, \quad \ldots\ldots(4)$$

their position being represented on the diagram by $x+iy$ in each case, the values of their corresponding simple coordinates being

$$\left.\begin{aligned} x&=ka\cos\phi\pm b\cos\phi\sqrt{(k^2-1)}\\ y&=kb\sin\phi\pm a\sin\phi\sqrt{(k^2-1)}\end{aligned}\right\}. \quad \ldots\ldots(5$$

If we now eliminate k, we find for the locus of the points of intersection of the series of parallel chords with the ellipse, the curve

$$x^2 \sec^2 \phi - y^2 \operatorname{cosec}^2 \phi = a^2 - b^2. \quad \ldots\ldots(6)$$

This is a confocal hyperbola, passing through the point $(a \cos \phi, b \sin \phi)$ which is the point of contact of (3) with (1), and which we may call the point P.

The asymptotes to this hyperbola make the angle ϕ with the axis of x, so that they pass through the points where the auxiliary circle of the ellipse is cut by the ordinate at P.

The mid-points of the pairs of conjugates lie on the diameter

$$bx \sin \phi = ay \cos \phi, \quad \ldots\ldots(7)$$

i.e. on the diameter passing through P.

The line through the mid-point M of any one of such pairs drawn perpendicular to the system of parallel chords will cut the hyperbola in the required pair of points. The equation of this line is

$$ax \sec \phi - by \operatorname{cosec} \phi = k(a^2 - b^2). \quad \ldots\ldots(8)$$

The pole of (8) with regard to the hyperbola is

$$\left(\frac{a \cos \phi}{k}, \frac{b \sin \phi}{k}\right),$$

which is also the pole of the line (2) with regard to the ellipse. We are thus led to the very interesting theorem that the tangents to the hyperbola drawn from the representative points (5) where the given chord (2) cuts the given ellipse intersect in the pole of the chord with respect to the ellipse.

Consequently not only do real points on the hyperbola represent imaginary points on the ellipse, but real tangents to the hyperbola represent imaginary tangents to the ellipse.

There are other interesting things connected with pairs of conjugate imaginary points on a conic, but the above are sufficient to show that the recognition of the right of imaginary points to be elevated to the dignity of plottable points on the plane will be rewarded by interesting developments. The main object of this paper is to plead that right.

Some of the developments are likely to be important, as evidently the matter is closely connected with the theory of orthogonal curves.

The connection of the above with the Argand diagram is too obvious to need pointing out. It shows that the method of plotting by rectangular coordinates is an extension of the Argand diagram to two variables. It has many interesting applications in the theory of equations.

A. Lodge.

Cooper's Hill.

AN ELEMENTARY INTRODUCTION TO THE INFINITESIMAL GEOMETRY OF SURFACES.

The purpose of the present note is to show how the student who is acquainted with the methods only of using and transforming rectangular axes is already in a position to determine the geodesic curvature and torsion of any curve traced on a surface, and in particular to prove Gauss' formula for the product of the principal curvatures in terms of the geodesic curvatures of the lines of curvature.

If the axes of x and y are turned in their plane about the origin through an angle θ, the formulae of transformation of coordinates are

$$x = x' \cos \theta - y' \sin \theta,$$
$$y = x' \sin \theta + y' \cos \theta,$$

leading to $ax^2+2hxy+by^2=a'x'^2+2h'x'y'+b'y'^2,$

where
$$a'=a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta,$$
$$h'=(b-a)\cos\theta\sin\theta+h(\cos^2\theta-\sin^2\theta),$$
$$b'=a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta,$$

so the term in $x'y'$ disappears when the angle of rotation is given by

$$\tan 2\theta=2h/(a-b).$$

When θ is so small that θ^2 may be neglected, these formulae become

$$x=x'-y'\theta,$$
$$y=x'\theta+y'.$$

The change of coordinates due to rotating the frame of reference through an angle θ_1 about the axis of x, and then through an angle θ_2 about the axis of y', when squares of the angles are neglected, is found by combining the transformations

$$\left.\begin{aligned} x&=x'\\ y&=y'-z'\theta_1\\ z&=z'+y'\theta_1 \end{aligned}\right\} \quad\text{and}\quad \left.\begin{aligned} y'&=y''\\ z'&=z''-x''\theta_2\\ x'&=x''+z''\theta_2 \end{aligned}\right\},$$

which lead to

$$z=z''-x''\theta_2+y''\theta_1,$$

showing that the order of the rotations is immaterial. Hence the plane $z''=0$ is, when referred to the original axes,

$$z+x\theta_2-y\theta_1=0.$$

We define the *normal curvature* of a curve traced on a surface as the component rate at which the tangent plane turns about the line in itself perpendicular to the tangent line to the curve, and the *geodesic torsion* as the component rate at which the tangent plane turns about the tangent line, the point of contact being supposed to describe the curve at unit rate.

Let the equation of a surface be expressed in the form

$$z=u_2+u_3+u_4+\ldots,$$

where $2u_2\equiv ax^2+2hxy+by^2,$

and u_3, u_4, ... are homogeneous polynomials in x, y of degrees indicated by their suffixes, so that the origin is on the surface, and the axis of z is normal to it.

Change the origin to $(\delta s_1, 0, 0)$ and neglect the square of δs_1; the new equation is

$$z-\frac{\partial u_2}{\partial x}\delta s_1=u_2+\frac{\partial u_3}{\partial x}\delta s_1+u_3+\frac{\partial u_4}{\partial x}\delta s_1+\ldots,$$

showing that the tangent plane at the new origin is

$$z-(ax+hy)\delta s_1=0.$$

On comparing this with

$$z+x\theta_2-y\theta_1=0,$$

we see that the tangent plane has undergone rotations

$$\theta_1=h\delta s_1,$$
$$\theta_2=-a\delta s_1,$$

so, without paying attention to sign, we may say that a is the normal curvature and h the geodesic torsion of the section by $y=0$. Similarly, for a

displacement δs_2 of the point of contact along the section by $x=0$, the tangent plane undergoes rotations $b\delta s_2$ and $-h\delta s_2$. We infer that *any* two curves on a surface which cut at right angles have the same (numerical) geodesic torsion.

Since by turning the axes of x and y it is possible to get rid of the term in xy, we infer that there are two curves through any point of the surface which have zero geodesic torsion; this may be taken to be the definition of *lines of curvature*, and when the term in xy is absent, the coefficients of x^2 and y^2 are called the *principal curvatures*.

The familiar forms for normal curvature and geodesic torsion are found simply by turning the axes of x and y through an angle θ. If then the equation

$$2z=\kappa_1x^2+\kappa_2y^2+u_3+\dots$$

becomes

$$2z=ax^2+2hxy+by^2+\dots,$$

we have

$$a=\kappa_1\cos^2\theta+\kappa_2\sin^2\theta,$$

giving Euler's formula for normal curvature, and

$$h=(\kappa_2-\kappa_1)\cos\theta\sin\theta,$$

giving the torsion of any geodesic.

In future we shall take $a=\kappa_1$, $h=0$, $b=\kappa_2$, so that the standard form of equation is

$$2z=\kappa_1x^2+\kappa_2y^2+u_3+u_4+\dots,$$

where

$$3u_3\equiv Ax^3+3Bx^2y+3Cxy^2+Dy^3,$$

$$12u_4\equiv\alpha x^4+4\beta x^3y+6\gamma x^2y^2+4\delta xy^3+\epsilon y^4.$$

In order to find the rates of change of the principal curvatures we transfer the origin to $(\delta s_1, 0, 0)$, and then turn the axes until the equation is again reduced to standard form; throughout the work squares of displacements are neglected. In the equation

$$2(z-\kappa_1x\delta s_1)=\kappa_1x^2+\kappa_2y^2+\frac{\partial u_3}{\partial x}\delta s_1+\dots$$

we must replace

$$z \text{ by } z+\kappa_1x\delta s_1$$

and

$$x \text{ by } x-\kappa_1z\delta s_1,$$

$$=x-\tfrac{1}{2}\kappa_1(\kappa_1x^2+\kappa_2y^2)\delta s_1+\dots.$$

This does not affect the quadratic terms, which are

$$(\kappa_1+A\delta s_1)x^2+2B\delta s_1xy+(\kappa_2+C\delta s_1)y^2.$$

To remove the middle term turn the axes through the small angle $B\delta s_1/(\kappa_1-\kappa_2)$, thereby leaving the coefficients of x^2 and y^2 unaltered. The equation is now of the standard form and we may indicate the changes which the coefficients have undergone by the symbol δ. Thus

$$\delta\kappa_1=A\delta s_1,\quad \delta\kappa_2=C\delta s_1,$$

that is to say, the coefficients A and C of the original equation are the rates of change of the principal curvatures along the first line of curvature. Similarly we may write $B=\delta\kappa_1/\delta s_2$, $D=\delta\kappa_2/\delta s_2$. It must carefully be noticed that the symbol δ is not the same as the symbol of differentiation, for although δs_1, δs_2 are elements of length, they are not perfect differentials of functions of the coordinates, and so the integrated forms s_1, s_2 have no definite meaning; much less therefore can κ_1, κ_2 be regarded as functions of s_1 and s_2.

We have seen that when the origin moves along a line of curvature and the axis of x is tangent to it, the axis of z being normal to the surface, then

the frame turns about the z-axis at a rate $B/(\kappa_1-\kappa_2)$, $=g_1$ say. Then g_1 is called the *geodesic curvature* of the line of curvature, and is positive when the turning is from x towards y. Similarly $C/(\kappa_2-\kappa_1)$, $=g_2$ say, is the geodesic curvature of the other line of curvature and is positive when the turning is from y towards x. In order to find Gauss' expression for $\kappa_1\kappa_2$ in terms of g_1, g_2 and their rates of change, we require to know $\delta C/\delta s_1$. In order to bring back the equation

$$2(z-\kappa_1 x\delta s_1)=\kappa_1 x^2+\kappa_2 y^2+\frac{\partial u_3}{\partial x}\delta s_1+u_3+\frac{\partial u_4}{\partial x}\delta s_1+\ldots$$

to standard form we have, as was shown above, to replace

$$\begin{aligned} x &\text{ by } x-\tfrac{1}{2}\kappa_1(\kappa_1 x^2+\kappa_2 y^2)\delta s_1-yg_1\delta s_1,\\ y &\text{ by } y+xg_1\delta s_1,\\ z &\text{ by } z+\kappa_1 x\delta s_1, \end{aligned}$$

and then the coefficient of xy^2 arises from the terms

$$\kappa_1 x^2+Bx^2y+(C+\gamma\delta s_1)xy^2+\tfrac{1}{3}Dy^3,$$

and is

$$-\kappa_1^2\kappa_2\delta s_1-2Bg_1\delta s_1+C+\gamma\delta s_1+Dg_1\delta s_1.$$

Accordingly we write

$$\delta C/\delta s_1=\gamma-\kappa_1^2\kappa_2+(BD-2B^2)/(\kappa_1-\kappa_2).$$

Similarly

$$\delta B/\delta s_2=\gamma-\kappa_1\kappa_2^2+(CA-2C^2)/(\kappa_2-\kappa_1),$$

whence

$$\begin{aligned} \delta g_1/\delta s_2+\delta g_2/\delta s_1&=(\delta B/\delta s_2-\delta C/\delta s_1)/(\kappa_1-\kappa_2)\\ &\quad-\{B(B-D)+C(C-A)\}/(\kappa_1-\kappa_2)^2\\ &=\kappa_1\kappa_2+(B^2+C^2)/(\kappa_1-\kappa_2)^2;\\ \therefore\ \kappa_1\kappa_2&=\delta g_1/\delta s_2+\delta g_2/\delta s_1-g_1^2-g_2^2, \end{aligned}$$

which is the formula required.

Since squares of small quantities are being neglected, the displacements δs_1, δs_2 may be superposed by addition. Hence when the origin is moved to $(\delta s_1, \delta s_2, 0)$ the frame must undergo a rotation

$$g_1\delta s_1-g_2\delta s_2$$

in order that the axes may be again tangents to lines of curvature. If then a curve makes a variable angle θ with the first line of curvature through any point of it, a frame having one axis always tangent to it and another normal to the surface must turn about the latter through an angle

$$g_1\delta s_1-g_2\delta s_2+\delta\theta,$$

from which we deduce that the rate of turning, which by definition is the geodesic curvature, is

$$g_1\cos\theta-g_2\sin\theta+d\theta/ds.$$

Having found the normal curvature, geodesic curvature and geodesic torsion of any curve on a surface, we are in a position to determine its curvature, torsion, and all other intrinsic properties.

Interesting examples of these methods are afforded by asymptotic curves and parabolic curves, but it is unnecessary here to pursue the details.

R. W. H. T. HUDSON.

LIVERPOOL, *November*, 1902.

PROOFS THAT THE ARITHMETIC MEAN IS GREATER THAN THE GEOMETRIC MEAN.

What follows is a condensed statement and classification of various proofs of the above well-known theorem.

For two positive quantities, a and b, the proof may be put as follows:

$$\frac{a+b}{2}-\sqrt{ab}=\frac{(\sqrt{a}-\sqrt{b})^2}{2}, \text{ which is } > 0.$$

$$\therefore \frac{a+b}{2}>\sqrt{ab}. \tag{1}$$

Otherwise, let the two quantities be denoted by $x+y$ and $x-y$, then their arithmetic mean x is $>$ their geometric mean $\sqrt{x^2-y^2}$.

Here we assume for the sake of simplicity, as we shall do throughout, that the given quantities are not all equal; otherwise the symbol $\not<$ would have to be used instead of $>$. We shall use A and G to denote the means.

To extend the theorem to n positive quantities $a, b, c, \ldots$, several methods are available, which may be classified as follows:

A. By substituting for the greatest and least another equal pair, which (α) leaves the A.M. unaltered and increases the G.M., or (β) leaves the G.M. unaltered and diminishes the A.M., but in both cases makes the A.M. and the G.M. continually approach equality.

B. By substituting for a and b, the greatest and least of the set, a pair of quantities having (α) the same G.M., or (β) the same A.M. as a and b, one of the pair being the mean of all; and so by a *finite* number of steps completing the proof.

C. By proving the theorem for $n=2^m$, and extending it to the general case by introducing 2^m-n equal quantities (α) each equal to G, or (β) each equal to A.

D. By mathematical induction, making the case for n quantities depend on that for $n-1$. Here also are two varieties, (α) and (β), in which A and G exchange rôles, and a third, (γ), depending on the inequality

$$\left(1+\frac{x}{n}\right)^n > \left(1+\frac{x}{n-1}\right)^{n-1}.$$

E. By showing that A and G are the extremes of a set of means of cognate type which are in descending order of magnitude.

Aα. Substitute for a and b, the greatest and least of the n given quantities, a pair of quantities each $=(a+b)/2$. We have thus a new set of n quantities whose sum is the same as that of the given set, but whose continued product is greater, since by (1),

$$\left(\frac{a+b}{2}\right)^2 > ab \quad \text{and} \quad \therefore \left(\frac{a+b}{2}\right)^2 . cde \ldots > abcde \ldots .$$

By taking a sufficient number of such steps we ultimately get a set having the same sum as $a, b, c, \ldots$ but a greater product, the members of which are as nearly equal to one another, and therefore to their G.M., as we please. At the same time this G.M. approaches as nearly as we please to the A.M. which is the same for all the sets. Since the G.M. of the last set is $>$ that of the given set by a finite amount, it follows that the G.M. of the latter is less than its A.M.

Aβ. Substitute for a and b their G.M. $\sqrt{ab}$ taken twice, and we get a new set of n quantities having the same continued product as that of $a, b, c, \ldots$, but a smaller sum, since

$$2\sqrt{ab}-a-b=-(\sqrt{a}-\sqrt{b})^2, \text{ which is } < 0.$$

Taking a sufficient number of such steps, we ultimately get a set of numbers as nearly equal to one another, to their A.M., and to their G.M., as we please, and having the same product as $a, b, c, \ldots$, but a smaller sum. Hence we deduce that the A.M. of the original set is greater than their G.M.

In order to make the proofs Aα and Aβ complete, it would be necessary to show that by taking a sufficient number of steps in each case, we *can* make the numbers of the resulting set approach mutual equality as nearly as we please.

Bα. Assuming for the present that both the arithmetic and the geometric means of n quantities lie between b and a, the least and the greatest of them, G, the geometric mean, will lie between a and b. Let then $k=\frac{ab}{G}$, so that

$$a+b-(G+k)=a+b-G-\frac{ab}{G}=\frac{1}{G}(a-G)(G-b), \text{ which is } >0;$$

$$\therefore\ a+b>G+k.$$

Thus we have a new set of quantities having the same continued product as the given set but a smaller sum, and including at least one quantity equal to G. By a series of similar steps, not more than $n-1$ in number, we arrive at n quantities G, having a smaller sum than the given set, whose A.M. is therefore $>G$.

Bβ. Next let A be the arithmetic mean of the given numbers, and let $k=a+b-A$, where, as before, a and b stand for the greatest and least of the set considered. Then A lies between a and b. Again,

$$Ak-ab=A(a+b-A)-ab=(a-A)(A-b), \text{ which is } >0.$$

Thus taking A and k for a and b, we have a new set of quantities having the same sum, but a greater continued product, and including at least one quantity equal to A. By a number of similar steps, $n-1$ at most, we arrive at a set of n quantities, each equal to A, which have a greater continued product than $a, b, c, \ldots$, and whose G.M. is therefore $>G$. Hence $A>G$.

To prove the assumptions that A and G lie between a and b, we have only to note that

(2) $\quad na-nA=(a-a)+(a-b)+(a-c)+\ldots,$ which is >0, so that $a>A$.

(3) $\quad nA-nb=(a-b)+(b-b)+(c-b)+\ldots,$ which is >0, so that $A>b$.

(4) $\quad a^n\div G^n=a^n\div(abc\ldots)=\frac{a}{a}\cdot\frac{a}{b}\cdot\frac{a}{c}\ldots,$ which is >1, so that $a>G$.

(5) $\quad G^n\div b^n=(abc\ldots)\div b^n=\frac{a}{b}\cdot\frac{b}{b}\cdot\frac{c}{b}\ldots,$ which is >1, so that $G>b$.

C. To prove the theorem when $n=2^m$, observe that

$$\begin{aligned}
&a+b+c+\ldots-n(a\,.\,b\,.\,c\ldots)^{\frac{1}{n}}\\
&\quad=(a^{\frac{1}{2}}-b^{\frac{1}{2}})^2+(c^{\frac{1}{2}}-d^{\frac{1}{2}})^2+\ldots\\
&\qquad+2\{(a^{\frac{1}{4}}b^{\frac{1}{4}}-c^{\frac{1}{4}}d^{\frac{1}{4}})^2+(e^{\frac{1}{4}}f^{\frac{1}{4}}-g^{\frac{1}{4}}h^{\frac{1}{4}})^2+\ldots\}\\
&\qquad+2^2\{(a^{\frac{1}{8}}b^{\frac{1}{8}}c^{\frac{1}{8}}d^{\frac{1}{8}}-e^{\frac{1}{8}}f^{\frac{1}{8}}g^{\frac{1}{8}}h^{\frac{1}{8}})^2+\ldots\}\\
&\qquad+\ldots+\frac{n}{2}\{(ab\ldots)^{\frac{1}{n}}-(uv\ldots)^{\frac{1}{n}}\}, \text{ which is } >0.
\end{aligned}$$

Hence $\quad (a+b+c\ldots)/n>(abc\ldots)^{\frac{1}{n}}.$ Q.E.D.

Cα. Next suppose n not a power of 2, and let $n+p=2^m$.

$$a+b+c\ldots+pG > (n+p)(abc\ldots G^p)^{\frac{1}{n+p}}$$
$$> (n+p)(G^nG^p)^{\frac{1}{n+p}}$$
$$> (n+p)G.$$
$$\therefore\ a+b+c\ldots > nG,$$

whence
$$A > G.$$

Cβ. Again, $a+b+c+\ldots+pA > (n+p)(abc\ldots A^p)^{\frac{1}{n+p}}$.

$$\therefore\ nA+pA > (n+p)(G^nA^p)^{\frac{1}{n+p}}.$$
$$\therefore\ A^{n+p} > G^nA^p.$$
$$\therefore\ A > G.$$

Dα. Arrange $a_1a_2\ldots$ in ascending order, so that $a_n > G_{n-1}$ (see (4)), and let

$$G_{n-1} \equiv (a_1a_2\ldots a_{n-1})^{\frac{1}{n-1}} \quad \text{and} \quad G_n \equiv (a_1a_2\ldots a_n)^{\frac{1}{n}}.$$

Assuming
$$a_1+a_2+\ldots+a_{n-1} > (n-1)G_{n-1}.$$
$$\therefore\ a_1+a_2+\ldots+a_{n-1}+a_n > (n-1)G_{n-1}+a_n.$$
$$\therefore\ \frac{a_1+a_2+\ldots+a_n}{n} > G_{n-1}+\frac{a_n-G_{n-1}}{n}.$$
$$\therefore\ A_n^n > \left(G_{n-1}+\frac{a_n-G_{n-1}}{n}\right)^n$$
$$> G_{n-1}^n + n.\,G_{n-1}^{n-1}\times\frac{a_n-G_{n-1}}{n}$$

by the Binomial Theorem, since a_n-G_{n-1} is positive.

$$\therefore\ A_n^n > G_{n-1}^{n-1}a_n$$
$$> a_1a_2\ldots a_{n-1}a_n.$$
$$\therefore\ A_n > G_n.$$

Hence, starting from (1), by mathematical induction, $A > G$.

Dβ. Assume $A_{n-1} > G_{n-1}$. By (2) $a_n > A_{n-1}$.

$$\therefore\ A_n = \frac{\overline{n-1}A_{n-1}+a_n}{n} = A_{n-1}+\frac{a_n-A_{n-1}}{n}.$$
$$\therefore\ A_n^n > A_{n-1}^n + nA_{n-1}^{n-1}\times\frac{a_n-A_{n-1}}{n}$$
$$> a_nA_{n-1}^{n-1} > a_nG_{n-1}^{n-1} > G_n^n.$$
$$\therefore\ A_n > G_n.$$

As before, by mathematical induction, $A > G$.

Dγ. Thacker's proof depends on this lemma: if x is positive, and n a positive integer,

$$\left(1+\frac{x}{n}\right)^n > \left(1+\frac{x}{n-1}\right)^{n-1};$$

which is proved by expanding both sides by the Binomial Theorem, and observing that the rth term of the former expansion is $>$ that of the latter, since

$$\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\ \ldots\ \frac{n-r+2}{n} > \frac{n-1}{n-1}\cdot\frac{n-2}{n-1}\cdot\frac{n-3}{n-1}\ \ldots\ \frac{n-r+1}{n-1}.$$

Hence
$$A_n^n = a_1^n\left(1+\frac{a_1+a_2+\ldots+a_n-na_1}{na_1}\right)^n$$
$$> a_1^n\left(1+\frac{a_2+a_3+\ldots+a_n-\overline{n-1}\,a_1}{(n-1)a_1}\right)^{n-1} \quad \text{by the lemma,}$$
$$> a_1^n\left(\frac{a_2+a_3+\ldots+a_n}{(n-1)a_1}\right)^{n-1}$$
$$> a_1\left(\frac{a_2+a_3+\ldots+a_n}{n-1}\right)^{n-1}.$$

Thus, assuming $A > G$ for $n-1$ quantities, we have
$$A_n^n > a_1(a_2a_3\ldots a_n) > G_n^n.$$
$$\therefore\ A_n > G_n.$$

Thus the theorem follows by mathematical induction.

E. Let $N_r \equiv$ (sum of products of $a, b, c, \ldots$ taken r at a time) $\div$ (number of such products), then it is proved in the Theory of Equations that

(6) $$N_r^2 > N_{r-1}N_{r+1}. \qquad \therefore\ \frac{N_r}{N_{r-1}} > \frac{N_{r+1}}{N_r}.$$

Hence
$$\frac{N_1}{1} > \frac{N_2}{N_1} > \frac{N_3}{N_2} \ldots > \frac{N_r}{N_{r-1}} > \frac{N_{r+1}}{N_r}.$$
$$\therefore\ \left(\frac{N_{r+1}}{N_r}\right)^r < \frac{N_r}{N_{r-1}}\cdot\frac{N_{r-1}}{N_{r-2}}\cdots\frac{N}{1} < N_r.$$
$$\therefore\ N_{r+1}^r < N_r^{r+1}. \qquad \therefore\ N_{r+1}^{\frac{1}{r+1}} < N_r^{\frac{1}{r}}.$$

Hence

(7) $$N_1 > N_2^{\frac{1}{2}} > N_3^{\frac{1}{3}} \ldots > N_n^{\frac{1}{n}}.$$

But $$N_1 \equiv A_n \quad \text{and} \quad N_n^{\frac{1}{n}} \equiv G_n.$$

Hence $$A_n > G_n.$$

The following formula, recently arrived at by the writer, expresses in a form which shows that it is necessarily positive, the excess of the arithmetic over the geometric mean of n quantities, written for convenience in the form $a^n, b^n, c^n \ldots$.

$$\frac{1}{n}\Sigma a^n - abcd\ldots = \frac{1}{n(n-1)}\Sigma(a-b)(a^{n-1}-b^{n-1})$$
$$+\frac{1!}{n(n-1)(n-2)}\Sigma(a-b)(a^{n-2}-b^{n-2})c$$
$$+\frac{2!}{n(n-1)(n-2)(n-3)}\Sigma(a-b)(a^{n-3}-b^{n-3})cd$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\frac{(r-1)!\,(n-r-1)!}{n!}\Sigma(a-b)(a^{n-r}-b^{n-r})cd\ldots$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\frac{(n-2)!}{n!}\Sigma(a-b)^2cd\ldots$$

where each term is of course of degree n in $a, b, c, d\ldots$.

The following references, though not exhaustive, may be found useful:

$A\alpha$ is given by Todhunter and by Chrystal; $A\alpha$ and $A\beta$ are both given by C. Smith.

$B\alpha$ is due to G. H. Bryan (see the *Tutorial Algebra*), and $B\beta$ to G. E. Crawford (*Edin. Math. Soc. Proceedings*, XVIII., p. 2).

$C\beta$ is given in Todhunter's *Algebra*, and is due to Cauchy (*Analyse Algébrique*, p. 457); and $C\alpha$ is obviously suggested by it.

$D\beta$ is given in H. Weber's *Algebra*, and $D\alpha$ is suggested by it. $D\gamma$ is due to Thacker (*Camb. and Dub. Math. Journ.*, VI., p. 81). Another proof, of type D, but depending on a different fundamental inequality, is given by Chrystal (*Algebra*, XXIV., § 8).

E is given by Schlömilch in his *Zeitschrift* (III., p. 301), with a proof depending on Theory of Equations, and attributed to Fort. A proof of (7) by mathematical induction, not involving Theory of Equations, will be found in the *Edin. Math. Soc. Proceedings*, XIX., p. 36. R. F. MUIRHEAD.

MATHEMATICAL NOTES.

124. [D. 2. b.] *Addition Series.*

An Addition series is one in which $u_n = u_{n-1} + u_{n-2}$, the principal addition series being

1, 1, 2, 3, 5, 8, 13, 21, etc., ...A.

which was originally invented to obtain numbers by which to divide, practically, a line in extreme and mean ratio. In fact, in A.,

$$u_n = u_{n-1} + u_{n-2}, \text{ and } u_n u_{n-2} - u_{n-1}^2 = \pm 1,$$

which is as near an approximation to E. and M. ratio as is possible in integers.

Here, $$u_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\};$$

and, by induction, the sum of n terms $= u_{n+2} - 1$.

The following are properties of A. which can be proved by induction, or from the value of u_n.

If $M = u_n u_{n-3} - u_{n-1} u_{n-2}$, then $M = \pm 1$ (M in G.P. and A.P. respectively being 0 and $-2b^2$).

$$u_0 = 0; \text{ and } u_{-n} = \pm u_n.$$

The mn^{th} term is divisible by the n^{th}.

$$u_n = u_m u_{n-m+1} + u_{m-1} u_{n-m}.$$

$u_n^2 - u_{n-2}^2$ and $u_n^2 + u_{n+1}^2$ are each equal to a term of A.

One of the two $u_n^2 \pm u_m^2$ is always equal to the product of two terms of A.

The general Addition series can be expressed in terms of A., for it is:

a, b, $a+b$, $a+2b$, $2a+3b$, $3a+5b$, etc. .....G.A.

The sum of n terms of G.A. is $au_n + b(u_{n+1} - 1)$ where u_n is a term of A.

In G.A., $M = \pm(a^2 + ab - b^2)$; and $U_n U_{n-2} - U_{n-1}^2 = M$; where U_n is a term of G.A.

The sum of n terms is $U_{n+2} - U_2$; and the sum of n terms after the m^{th} is $U_{n+m+2} - U_{m+2}$.

Also $$U_{2n+1} = (1+1)^n U_1 U_2 U_3 \ldots U_{n+1}).$$

In the series $a,\ b,\ a+b,\ 2a+b,\ 3a+2b,\ 4a+3b$, etc.,

in which $U_n,\ U_{n+1},\ U_{n+2}$ is an A.P. if n is even, and a G.A. if n is odd,

$$M=\pm(2a^2-b^2), \text{ and } U_n=2U_{n-2}+U_{n-4}.$$

In the series composed of two G.A. placed alternately

$$a,\ x,\ b,\ y,\ a+b,\ x+y,\ a+2b,\ x+2y, \text{ etc.},$$

$M=$ alternately $\pm(ay-bx)$ and $\pm\{a+bx-by\}$; thus $M=\pm1$, in

$$2,\ 1,\ 3,\ 2,\ 5,\ 3,\ 8,\ 5,\ 13,\ 8, \text{ etc.}$$

M is, in turn, zero and $\pm$(a term of A.)^2 in the following series in which $u_1,\ u_4,\ u_7$, etc., is A.; and also $u_2,\ u_5,\ u_8$, etc.; and $u_3,\ u_6,\ u_9$, etc.; viz.:

$$1,\ 2,\ 3,\ 2,\ 3,\ 5,\ 3,\ 5,\ 8,\ 5,\ 8,\ 13, \text{ etc.}$$

It is seen from

$$1,\ r,\ r^2,\ r+r^2,\ (1+r)^2,\ (1+3r+2r^2), \text{ etc.},$$

that it is possible to have a series of integral functions in which $u_n,\ u_{n+1},\ u_{n+2}$ is a G.P. if n is odd, and a G.A. if n is even. W. H. LAVERTY.

125. [K. 20. d.] *Approximation to tan A.*

Up to 10° the ordinary formula, $\tan A^\circ = \cdot0175A$, gives close enough results for many purposes; but from 10° to 25° or so it is better to use

$$\tan A=\frac{A}{60}+\frac{A}{6000}+\left(\frac{A}{120}\right)^2.$$

Three examples are given.

$A=$	10°·8	16°·7	24°·6
60)	10·8	16·7	24·6
	·1800	·2783	·4100
	18	28	41
	81	196	420
$\tan A=$	·1899	·3007	·4561
From tables,	·1908	·3000	·4578

are seen to be the more accurate values.

The error is usually less than that in the last example, and as far as I have tried, the calculation gives the true tangent of some angles within a very few minutes of the given one.

An approximate reverse formula is

$$A=61\tan A-16\tan^2 A,$$

which may be applied to any angle up to 45° with fair exactness, the maximum error in the value of A so found being about a quarter degree.

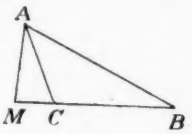
I have been told that the latter has already appeared in some book on Yachting. C. E. M'VICKER.

126. [V. a; K. 20. e.] *Note on the Teaching of "Solution of Triangles" in Trigonometry.*

The methods commonly described in text-books for the logarithmic solution of oblique-angled triangles do not permit of the subject being adequately studied until a knowledge has been acquired of the properties of multiple and submultiple angles, and the so-called semi-sum and semi-difference formulae of Trigonometry. The object of the present note is to show how a beginner may proceed directly to the logarithmic solution of oblique-angled triangles with no knowledge beyond what is involved in the definitions of the trigonometric functions, the use of logarithmic tables, together with a preliminary drilling in solution of right-angled triangles.

The cases where a side and two angles or two sides and an angle opposite one are given present no difficulty, as they are solved by the sine rule which can be easily proved. The remaining cases are those in which two sides and the included angle or three sides are given.

For the first of these cases let CA, CB be the two given sides, BCA being the included angle, and suppose CB is the greater of the two sides (so that we do not have to consider the possibility of B being obtuse).



Drop AM perpendicular on BC.

Then we have

if C be acute	if C be obtuse
$MA = b \sin C$,	$MA = b \sin (180° - C)$,
$CM = b \cos C$,	$CM = b \cos (180° - C)$,
$BM = a - CM$.	$BM = a + CM$.

$$\tan B = \frac{AM}{BM},$$

and
$$c = \frac{AM}{\sin B} \quad \text{or} \quad c = \frac{BM}{\cos B}.$$

Finally,
$$A = 180° - B - C.$$

For the second case let a be the greatest of the three sides (so that we do not have to consider the possibility of either B or C being obtuse), and suppose $b > c$.

Drop AM perpendicular on BC as before.

Then, by I. 47,
$$CM^2 - BM^2 = CA^2 - BA^2 = b^2 - c^2,$$

and
$$CM + BM = a.$$

Hence
$$\log (CM - BM) = \log (b + c) + \log (b - c) - \log a.$$

This finds CM and BM, and then the angles are given by the formulae

$$\cos B = \frac{BM}{c}, \quad \cos C = \frac{CM}{b}.$$

The following examples will show how simply and easily triangles can be solved by these methods with a book of four-figure tables.

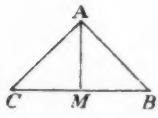
Ex. 1. $a = 684$, $b = 504$, $C = 94° 16'$.

Here C is obtuse, and $CM = b \cos 85° 44'$.

		Log.	Antilog.	
	b	2·7024		(1)
	$\cos (180° - C)$	8·8717		(2)
	$\sin (180° - C)$	9·9988		(3)
From (1), (2),	CM	1·5741	37·51	(4)
From (1), (3),	MA	2·7012	502·4	(5)
	BM	2·8582	721·51	(6)
From (5), (6),	$\tan B$	9·8430	$B = 34° 52'$	(7)
	$\sin B$	9·7577		(8)
From (5), (8),	AB	2·9441	$c = 879·2$	(9)

Finally, $A = 180° - 94° 16' - 34° 52' = 50° 52'$.

Ex. 2. $a=45{\cdot}2$, $b=32{\cdot}9$, $c=15{\cdot}4$.



	No.	Log.
b	32·9	
c	15·4	
$b+c$	48·3	1·6839
$b-c$	17·5	1·2430
b^2-c^2		2·9269
$CM+BM=a$	45·2	1·6551
$CM-BM$	18·70	1·2718
$2CM$	63·90	
$2BM$	26·50	

	No.	Log.		No.	Log.
CM	31·95	1·5045	BM	13·25	1·1222
$CA(=b)$	32·9	1·5172	$BA(=c)$	15·4	1·1875
$\cos C$		9·9873	$\cos B$		9·9347

$\therefore\ C=13^\circ\ 47'$ $\qquad B=30^\circ\ 38'$

Finally, $A=180^\circ-13^\circ\,47'-30^\circ\,38'=135^\circ\,35'$.

The solutions of these two examples by conventional methods would be as follows:

Ex. 1.

		No.	Log.
a		684	
b		504	
$a-b$		180	2·2553
$a+b$		1188	3·0748
			$\overline{1}{\cdot}1805$
$\cot\frac{1}{2}C$		$\cot 47^\circ\,8'$	9·9676
$\tan\frac{1}{2}(A-B)$			9·1481
$\frac{1}{2}(A-B)$	$=$	$8^\circ\,0'$	
$\frac{1}{2}(A+B)$	$=$	$42^\circ\,52'$	
A	$=$	$50^\circ\,52'$	
B	$=$	$34^\circ\,52'$	
$a+b$			3·0748
$\sin\frac{1}{2}C$			9·8651
$\cos\frac{1}{2}(A-B)$			12·9399
			9·9958
c		879·2	2·9441

Ex. 2.

	No.	Log.
a	45·2	
b	32·9	
c	15·4	
	2)93·5	
s	46·75	1·6693
$s-a$	1·55	0·1903
$s-b$	13·85	1·1415
$s-c$	31·35	1·4962
		$\overline{2}{\cdot}5305$
$(s-a)\div s$		·3557
$(s-c)\div(s-b)$		$\overline{2}{\cdot}8862$
$\tan^2\frac{1}{2}B$		$\overline{2}{\cdot}1748$
$\tan^2\frac{1}{2}C$		9·4437
$\tan\frac{1}{2}B$	$\tan 15^\circ\,30'$	9·0874
$\tan\frac{1}{2}C$	$\tan\ 6^\circ\,58'$	
B	$31^\circ\ 0'$	
C	$13^\circ\,56'$	

Finally, $A=180^\circ-31^\circ\,0'-13^\circ\,56'=135^\circ\,4'$.

A comparison of the alternative methods shows that there is no appreciable saving of labour in the use of the elaborate formulae commonly taught, and that by the use of the simpler methods here suggested the subject may be taught to beginners at an earlier stage than has been hitherto customary.

G. H. Bryan.

REVIEWS.

A Course of Modern Analysis. By E. T. Whittaker, M.A. Pp. xvi. 378. (Cambridge University Press. 1902.)

This work is sure of a favourable reception because it gives in a moderate compass an attractive account of some of the most valuable and interesting results of recent analysis. The first part deals mainly with infinite series, especially

power-series and Fourier expansions: it also includes the elements of complex integration and of the theory of residues. The second part comprises chapters on the gamma-function, Legendre functions, the hypergeometric series, Bessel functions, the equations of mathematical physics, and elliptic functions.

The author has secured brevity by adopting, on the whole, a deductive method of exposition. The advantages of this are shown very clearly in the chapter on the gamma-function. By taking Weierstrass's definition of $\Gamma(z)$ as an infinite product, the properties of the function are deduced with extreme simplicity, and although the historical order of development is almost reversed, we feel that the new treatment is really the true and natural one. The gamma-function is, in fact, the simplest case of a transcendent function with an infinite number of simple poles, and this property is made obvious by the new definition. (It is curious, by the bye, to see how very nearly Weierstrass was anticipated in this matter by Gauss.) Again, the new method has suggested interesting extensions to Alexeiewsky, Barnes, and others, so that the change of view has amply justified itself.

On the other hand, Mr. Whittaker's chapter on the hypergeometric series seems to me to illustrate the risk of becoming artificial, which is incurred by an author who sacrifices induction overmuch. Here, for instance, we have Papperitz's form of the differential equation satisfied by the most general form of Riemann's P-function set down without any explanation of the method by which it is obtained. For all the reader can tell, it might have dropped from the skies, or have been written on a wall by some spirit from another world. It has not entered into the plan of the treatise to give any general account of linear differential equations: but in this context it would have been possible to bring out the essential fact that the equation defines a function with three critical points, each associated with two indices. The suppression of this fundamental property obscures the real meaning of much that follows, and gives an air of hocus-pocus to the demonstrations. The principal vice of Cambridge methods and Cambridge text-books has always been the encouragement of students to take their mathematics ready-made, to assimilate facts without inquiring into their sources. By some of us, at least, it was hoped that the institution of Part II. of the Mathematical Tripos would help to counteract this tendency. Alas! the conscientious ingenuity of tutors and lecturers, aiming at immediate results, is likely to defeat the main purpose of the innovation, and produce a state of things worse than that under the old régime.

But to return to Mr. Whittaker's book. One very good feature is that the relations of the functions of Legendre, Bessel, etc., to the general hypergeometric function have been brought out. If Chaps. x., xi. had been transposed this might perhaps have been done even more effectively. Such generalisations as this are the direct result of recent function-theory, and help to lighten the burden with which the progress of mathematics appears to threaten us. Apart from special subtleties, which appeal to the adept, the general tendency of function-theory is towards simplicity and clearness. There is no reason, for instance, why a school-boy should not learn the proper expression for $\sin x$ as an infinite product, or the elementary theory of power-series for a complex variable. New ideas are not difficult simply because they are new: if there is something which I never heard of until I was fifty, it does not follow that no one else is to hear of it before reaching the same age. And why persist in putting before boys and girls ideas which are positively erroneous, and methods entirely out of date?

This same gain of simplicity is illustrated by Mr. Whittaker's chapters on elliptic functions. It is not sufficiently shown why the ratio of the periods should be non-real; and, personally, I should prefer the omission of proofs of the addition-theorem which are not straightforward applications of Abel's method. But with these exceptions the discussion, as far as it goes, is very clear; and apart from the theories of transformation and complex multiplication (the omission of which is natural enough), the reader will find all the facts he is likely to require. One reflection suggested by reading this and other parts of the book is that the reputation of Cauchy and Liouville are likely to be enhanced as time goes on. It seems to me that even yet their contributions to science are insufficiently recognised. In this connection it may be remarked that Mr. Whittaker gives, besides Dirichlet's proof of the Fourier expansions, Cauchy's second demonstration (published in 1827) with some necessary amendments.

In some respects, I think, this treatise improves as it goes on. The first two or three chapters are neither so agreeably nor so accurately written as those which follow, and there is an occasional want of proportion. For instance (p. 12) "Series whose convergence *is due to* the convergence of the series formed by the moduli of their terms . . . are called absolutely convergent series" (the italics are mine): again, it is assumed (p. 13) that u^{n+1} tends to zero for $n=\infty$ when $|u|<1$, although facts quite as obvious, or more so, are proved in detail; on p. 19 the statement

"As n increases, v_{n+1}/v_n will therefore tend to the limit
$1-(1+\frac{1}{2}c)/n$"

is not satisfactory; while the second part of Art. 6 seems to me to obscure a very simple matter. Finally, I confess that I cannot follow all the argument of Arts. 84, 85: this is probably my own fault.

Of casual slips and misprints there are apparently few. It is, of course, wrong to say that $(z-1)!$ is a polynomial (p. 209); on p. 24 $(u+p)$ should be $(n+p)$: p. 46 for "excluded" read "included"; p. 201 for "Alexerewsky" read "Alexeiewsky." G. B. MATHEWS.

Practical Exercises in Geometry. By W. D. EGGAR, M.B. Macmillan & Co.

A New Geometry for Beginners. By R. ROBERTS, B.Sc. Blackie & Son.

Geometry. By S. O. ANDREW, M.A. John Murray.

Elements of Geometry. By R. LACHLAN, Sc.D. and W. C. FLETCHER, M.A. Edward Arnold.

Elementary Geometry. By W. M. BAKER, M.A. and A. A. BOURNE, M.A. G. Bell & Sons.

Plane Geometry. By T. PETCH, B.A. Edward Arnold.

Theoretical Geometry for Beginners. By C. H. ALLCOCK. Macmillan & Co.

Works on Geometry, whose appearance is due to alterations in the regulations of various examining bodies, continue to issue from the press. They may be divided roughly into two classes—(1) those which devote themselves to a course of Experimental Geometry, intended to precede and prepare for the future course of Deductive Geometry; (2) those intended to supply that future course, and replace the editions of Euclid's Elements, modernised or otherwise, now current in the schools.

There seems more immediate need of the former, the experimental courses throwing a great strain on teachers who have to deal with many other subjects besides Geometry, and have neither the time nor the inclination to develop a systematic course of experimental work. Excellent as are the little works by Mault, Bert, and Spencer, something more is wanted, and we welcome heartily Mr. Eggar's book as one that should supply a widely-felt want. It can be used both by those who are preparing their pupils for, and by those who are taking their pupils through, a deductive course. We hope it will be used largely by teachers of both categories, for we hold that experimental methods should *accompany*, as well as *precede*, deduction. How much vividness a theorem in loci gains, for instance, by such experimental treatment as that given on pp. 170, 186! The book is well got up; the figures are effectively drawn on an ample scale. We see everywhere signs of a teacher whose heart is in his work, and whose efforts, we feel assured, will help to kindle enthusiasm, not only among those who are under his direct personal influence, but among those also who are trained after the model he has set.

Mr. Roberts' work is of an intermediate character. Though without special experience of their needs, we imagine it excellently suited for technical schools. It is a rapid course through the essentials of Geometry. Theorems and problems seem judiciously chosen, and calculated to interest the student and lead to further studies and applications. We can best explain how condensed the scheme of treatment is by the statement that out of the 87 pages of which it consists, 5 are devoted to graphs and curve tracing, and that in the other 82 the author treats the subject matter of the ordinary school Euclid both theoretically and practically. The diagrams are numerous and well drawn; the five pages on curve tracing are

illustrated by 8 full-page diagrams, whose black lines stand out effectively on the tawny yellow of the millimetre ruled paper. We fancy it will be found useful by many teachers, besides those who use it as a class book. We could find fault with some of the definitions, but we could find the same, or greater, with those of a preceding writer, whose merits have been held to cover them. Mr. Andrew's work is of a similar character, his course being more extensive. In 178 pages he not only gets through a course of Plane and Solid Geometry, but also treats of Graphs, Mensuration, Trigonometry, and Descriptive Geometry. It seems likely to be very useful to teachers. The course is well chosen, the diagrams and type excellent. The remaining four books belong to the second class, and differ from each other chiefly in the extent of the course given, and in fulness of treatment. All are good examples of typographical excellence, which is more common now than in the days of Potts and Todhunter. Any one of them in the hands of a good teacher would be found an excellent basis for a course of lessons. Messrs. Lachlan & Fletcher in 204 pages give a course of the Elements of Plane Geometry, with two useful sections dealing with Geometrical Analysis and Maxima and Minima. They evade the difficulties of parallels by use of the "direction." Ratio and Proportion, and the Circle precede Areas. There is a numerous set of exercises, and illustrations from Mensuration and Trigonometry are appropriately given. Messrs. Baker and Bourne in 272 pages get through the essentials of Euclid's four books. The treatment on the whole is conservative. "Playfair's Axiom" replaces Euclid's celebrated postulate on parallels. Algebraical proofs are given of the principal relations between the segments of a line, and Arithmetical examples given throughout. Nine pages are devoted to Graphs.

Mr. Petch in 112 pages gets through a serviceable course, extending to Similarity. Mr. Allcock devotes 135 pages chiefly to the subject matter of Euclid's First Book. There are sections on the numerical treatment of Area, Analyses and Loci Parallel are treated by Playfair's Axiom. A convenient list of Definitions, Axioms, and Postulates is given at the end.

The Essentials of Plane and Solid Geometry, by WEBSTER WELLS, S.B. (D. C. Heath & Co., Boston.)

Mr. Wells has arranged his *Essentials* on much the same plan as the shorter English treatises noticed above, but extends to spherical Geometry. It seems an excellent treatise. Its scope is much the same as that of the revised edition of Beman and Smith's *Plane and Solid Geometry*. We do not know if there is any English work published in England of the same range. Doubtless such will not be long in making their appearance. English authors may well take the *designs* and English firms the *execution* of diagrams in these books for models. There is a dainty grace about them which ours seem to lack.

High School and Academic Algebra, by LOUIS PARKER JOCELYN. (Butler, Sheldon & Co., New York.)

This seems a clearly written and carefully arranged treatise. It extends to Permutations and Combinations and devotes sections to Logarithms (Practical), Undetermined Coefficients, Partial Fractions, and Determinants. There are numerous model solutions and large collections of exercises. The author lays great stress on 'checking' of solutions.

Advanced Algebra and Geometry, by WILLIAM BRIGGS, LL.D., M.A. (W. B. Clive.)

This is best characterised by its sub-title, as the Algebra and Geometry required for the Syllabus in Advanced Mathematics of the London University Matriculation Examination. It can be heartily recommended for its purpose.

EDWARD M. LANGLEY.

A Graphic Method for solving certain Questions in Arithmetic or Algebra, by G. L. VOSE. 2nd Edition. Pp. 62. 1902. (Van Nostrand, New York.)

The methods described in this little brochure were originally devised for railway purposes. How difficult such problems can be, may be guessed from a glance at train connections on a time-table. Perhaps some of our readers may like to amuse themselves by constructing a graphical solution of the following

simple case. A man walks round an island at the rate of five miles an hour, and at the same time another starts at such a rate as to take him round its ten mile circumference in three and one-third hours. When will the first pass the second, and when will he pass him the second time? We have found that boys are fond of these games.

Annuaire pour l'An 1903. Published by the Bureau of Longitudes. Pp. 668, 140. 1903. 1 fr. 50 c. (Gauthier-Villars.)

This little volume maintains its reputation for containing more matter than probably any other book in the world. So vast is the amount of astronomical, physical, geographical, and statistical information it contains, that, as the preface states, "il est claire qu'on ne saurait aller plus loin." Accordingly a re-arrangement of the contents has become imperative. Tables that never or seldom vary will not be yearly repeated, but the reader may count on getting from any two consecutive yearly numbers the tables he requires. Apart from the collection of tables, the present volume contains essays on shooting stars and comets by M. Radau; on Science and Poetry by the veteran astronomer, M. Janssen; the funeral orations on MM. Cornu and Faye; and an account of M. Janssen's work in 1902 at the observatory on Mont Blanc. The editors of the *Annuaire* are MM. Janssen, Lippman, Loewy, and last but not least, M. Poincaré. We cannot refrain from quoting an anecdote told by M. Janssen, expressing the awe which is felt by minds of a higher order on being first confronted with a striking instance of the laws which govern the universe. When Janssen was explaining to his pupil Gounod the laws of Kepler, "Gounod s'écria: 'Ah! que c'est beau!' et des larmes lui montèrent aux yeux."

Récréations Arithmétiques, by E. FOURREY. 2nd Edition. 1901. (Nony.)

This interesting collection of amusing arithmetical conundrums will make a capital birthday book for a boy in whom we wish to cultivate a taste for enquiry. They require but an elementary knowledge of the rules of arithmetic. The first part deals with the operations of the art, progressions, polygonal numbers, squares, cubes, divisors, and the simpler properties of numbers. The applications comprise such exercises as finding the day of the week corresponding to a given date, card tricks, the "think of a number" type, and oh! l'éternel féminin! the "three jealous husbands crossing a river with their wives, the boat available holding but two at a time"; and the like. The third part consists of an admirable introduction to the mysteries of squares, magic, diabolical and hypermagical, magic cubes and magic circles.

I. Tre Problemi Classici degli Antichi:—Prob. I. **La Quadrature del Cerchio,** by Prof. BELLINO CARRARA S.J. Pp. 168. Reprinted from the *Rivista di Fisica, Matematica e Scienze Naturali.* 1902. (Fusi, Pavia.)

In this pamphlet Prof. Carrara takes as the first of the three great problems of antiquity that of the squaring of the circle, and gives the reader a full historical and critical analysis of the attempts that have been made to solve it. To those who can read Italian, this will be found a most convenient summary. The style is easy, and references are copious. The various stages in the attempt to determine the ratio of the diameter to the circumference are clearly sketched, and the influence of the growth of modern analysis is skilfully and lucidly traced. To many the most interesting portion of Prof. Carrara's labours will be the pages in which he sets forth the progress of investigation from the advent of Hermite's *Sur la Fonction exponentielle,* 1873, and the gradual and successive simplification of the researches of that writer and Lindemann by Weierstrass, Hilbert, Hurwitz, and Gordan. Gordan's treatment is, in fact, quite elementary. It may be found in an English dress in chaps. iii. and iv. of Part II. of Klein's *Famous Problems,* (Ginn and Co., 1897.) We look forward with interest to the appearance of the companion volumes which will deal with the Delian problem and that of the trisection of an arbitrary angle.

The Schoolmasters' Year Book and Directory. 1903. (Swan Sonnenschein.)

We break our general rule in drawing the attention of readers of the *Gazette* to the appearance of this extremely handy volume. It contains lists with necessary

details of all the educational societies and organisations in the country, together with a vast amount of information useful to teachers which cannot be found within the covers of any other publication with which we are acquainted. Part III. contains a series of articles and reviews, including a notable article by Prof. Minchin on "The Reform of Mathematical Teaching." We cordially greet the advent of the *Year Book*. As for the *Directory*, we cannot agree with the Head Master who thought "the whole idea absolutely rotten." On the contrary, it will play its part in the raising the status of the profession.

A Treatise on Differential Equations. By A. R. FORSYTH. Third Edition. Pp. 511. 14s. 1903. (Macmillan.)

Good wine needs no bush. Professor Forsyth's treatise on Differential Equations remains a classic. It is the most lucid, accurate, and exhaustive exposition of the subject in our language. It is a matter of pride to English mathematicians to know that it had been translated into both German and Italian. Students may be interested to know that Dr. Maser's translation, published at Brunswick in 1889, contains 256 pages of solutions to the questions. To be up to date, Dr. Maser will have to furnish solutions to between two to three hundred more questions, which are appended to the new edition. More than eighty pages of additional matter are included in the volume before us. A sketch is given of Runge's method for the numerical solution of ordinary differentials, limited to insoluble equations of the first order (*Math. Ann.* xlvi., 1895, pp. 168-178). We also find an outline of the method for the integration of linear equations in series which is connected with the name of Frobenius (Crelle, lxxvi., 1873, pp. 214-224, and Forsyth's *Theory of Diff. Eq.*, vol. iv., pp. 78 *et seq.*). This, again, applying to equations of any order, is confined to those of the second order, with special applications to Bessel's equations. A welcome addition is the introduction to Jacobi's *Theory of Multipliers*, pp. 319-328.

PROBLEMS.

481. [L¹. 10. d.] PQ is a chord of a parabola meeting the axis in a fixed point H so that $AH=8AS$, and $P'Q'$ its projection on the tangent at the vertex: shew that the locus of the intersection of PQ'. $P'Q$ is a circle.

E. N. BARISIEN.

482. [J. 1. d.] Each sheet of a book is folded n times so as to make 2^{n+1} pages. Investigate the way in which the different pages must be arranged in the forms when the type is put into pages. G. H. BRYAN.

483. [L¹. 4. c.] The tangents to an ellipse at Q and R intersect at a point P on a coaxial ellipse whose axes are respectively n times those of the former curve. Find the locus of the centroid of the triangle PQR.

A. F. VAN DER HEYDEN.

484. [K. 22. d.] The pole of a small circle on the earth is in latitude l and longitude λ and the angular radius of the circle is α. Find the equation of its stereographic projection (the eye being in the equator in longitude 180°).

C. S. JACKSON.

485. [M¹. 3. 1. β.] P' is the centre of curvature at P, a point on

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=(2a)^{\frac{2}{3}}.$$

Find the locus of the intersection of OP' with the tangent at P, O being the origin. V. RETALI.

486. [I. 3.] Are there any values of p for which $2^{p-1}-1$ is divisible by p^2?

D. O. S.

487. [R. 7. f.] A particle is tied to a fixed point by a string whose natural length is a, and whose extension is proportional to the tension; when the particle is in equilibrium with the string vertical the extension is c: prove

that when the string revolves as a conical pendulum about the vertical, making n revolutions per second, the extension is

$$\frac{4\pi^2n^2ac}{(g-4\pi^2n^2c)}.$$ (C.)

488. [R. 4. a. β.] A rough uniform rod (coefficient of friction $\frac{1}{2}$) rests over the edge of a cylindrical bucket of diameter 9 inches, with its lower end against the vertical side of the bucket, and when it is on the point of sliding down it makes with the vertical an angle $\tan^{-1}\frac{3}{4}$: shew that its length is 40 inches. (C.)

489. [R. b. γ.] Particles are projected with given velocity from a point at a distance a from a vertical wall so as just to clear the horizontal top of the wall: prove that their points of impact on a vertical plane parallel to the wall and at a distance $2a$ from the point of projection lie on a circle. (C.)

490. [R. 4. c.] A framework formed of five light bars AB, BC, CD, DA, AC, $ABCD$ being a rectangle, is in equilibrium under the action of forces at the angles whose components in directions AB, BC respectively are X_1, X_2, X_3, X_4, and Y_1, Y_2, Y_3, Y_4. Shew that the stress in the diagonal bar AC is

$$(X_3+X_4)\frac{AB}{AC}+(Y_2+Y_3)\frac{BC}{AC}.$$ Melbourne, 1901.

SOLUTIONS.

338. [D. 2. b. β.] *If* $\theta+\phi+\psi=4\pi$,

$$\sum_0^\infty\left[(-1)^r\frac{\Sigma\theta^{2r}}{\lfloor 2r}\right]+1$$
$$=4\prod_0^\infty\left[1-\frac{\Sigma\theta^2}{\pi^2}\cdot\frac{1}{(2r+1)^2}+\frac{\Sigma\theta^2\phi^2}{\pi^4}\cdot\frac{1}{(2r+1)^4}-\frac{\theta^2\phi^2\psi^2}{\pi^6}\cdot\frac{1}{(2r+1)^6}\right].$$

L. Wallis.

Solution by Anon.

$$\sum_0^\infty\left\{(-1)^r\frac{\theta^{2r}+\phi^{2r}+\psi^{2r}}{\lfloor 2r}\right\}+1$$
$$=1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\frac{\theta^6}{\lfloor 6}+\ldots$$
$$+1-\frac{\phi^2}{\lfloor 2}+\ldots$$
$$+1-\frac{\psi^2}{\lfloor 2}+\ldots+1$$
$$=\Sigma\cos\psi+1$$
$$=2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}+2\cos^2\frac{\psi}{2}$$
$$=2\cos\frac{\psi}{2}\left\{\cos\frac{\theta+\phi}{2}+\cos\frac{\theta-\phi}{2}\right\}$$
$$=4\Pi\cos\frac{\psi}{2}$$
$$=4\prod_0^\infty\left(1-\frac{\theta^2}{\pi^2(2r+1)^2}\right)\Pi\left(1-\frac{\phi^2}{\pi^2(2r+1)^2}\right)\Pi\left(1-\frac{\psi^2}{\pi^2(2r+1)^2}\right)$$
$$=4\prod_0^\infty\left\{1-\frac{\Sigma\theta^2}{(2r+1)^2\pi^2}+\frac{\Sigma\theta^2\phi^2}{(2r+1)^4\pi^4}-\frac{\theta^2\phi^2\psi^2}{(2r+1)^6\pi^6}\right\}.$$

382. [D. 2. d.] *Prove that* $(\Sigma p_n)^2+(\Sigma p_n+1)^2=(p_{2n}+q_{2n})^2=q^2_{2n+1}$, *where* Σp_n *signifies the sum of the even convergents of* $p^2-2q^2=\pm 1$.

[Ex. $n=3$. $119^2+120^2=(99+70)^2=169^2$.]

R. W. D. CHRISTIE.

Solution.

$$\sqrt{2}=1+\frac{1}{2+}\,\frac{1}{2+}\cdots;$$

$$\therefore\ 1+\frac{p_{n+1}}{q_{n+1}}=2+\frac{1}{2+}\,\frac{1}{2+}\cdots\quad(n+1\text{ terms})$$

$$=2+\frac{1}{1+}\,\frac{p_n}{q_n}$$

$$=\frac{2p_n+3q_n}{p_n+q_n}.$$

Hence $$q_{n+1}=p_n+q_n.$$

Again, $$p_{2n+1}=2p_{2n}+p_{2n-1}$$

$$=2p_{2n}+2p_{2n-2}+p_{2n-3}$$

$$\cdots\cdots\cdots\cdots$$

$$=2\Sigma p_n+p_1;$$

$$\therefore\ \Sigma p_n=\tfrac{1}{2}(p_{2n+1}-1),$$

and $$(\Sigma p_n)^2+(\Sigma p_n+1)^2=\tfrac{1}{2}(p^2{}_{2n+1}+1)$$

$$=q^2{}_{2n+1}=(p_{2n}+q_{2n})^2.$$

410. [L¹. 4. c; M¹. 3. b.] *On the tangent at* M *to an ellipse is measured off a length* MN *equal to the distance of the tangent from the centre of the ellipse: prove that the locus of* N *is a unicursal whose area is* $\frac{\pi}{2}(a+b)^2$.

E. N. BARISIEN.

Solution by PROPOSER *and* H. G. BELL.

If ϕ be eccentric angle of M,

$$MN=p=\frac{ab}{\sqrt{b^2\cos^2\phi+a^2\sin^2\phi}},$$

and makes with transverse axis an angle $\omega=\tan^{-1}\left(\frac{b}{a}\cot\phi\right)$.

Hence the coordinates of N are

$$a\cos\phi+\frac{a^2b\sin\phi}{a^2\sin^2\phi+b^2\cos^2\phi},\quad b\sin\phi-\frac{ab^2\cos\phi}{a^2\sin^2\phi+b^2\cos^2\phi},$$

and the locus of N is unicursal.

The area outside ellipse

$$=\int_0^{2\pi}\tfrac{1}{2}p^2d\omega=\tfrac{1}{2}\int_0^{2\pi}(a^2\cos^2\omega+b^2\sin^2\omega)\,d\omega$$

$$=\frac{\pi}{2}(a^2+b^2),$$

and the whole area is $\frac{\pi}{2}(a^2+b^2)+\pi ab=\frac{\pi}{2}(a+b)^2$.

412. [L¹. 2; 14. a.] *E is a conic circumscribing the triangle ABC, and its centre O lies in a fixed straight line: shew that the poles of the sides of ABC with respect to E lie on three conics which cut the corresponding sides in points on a fourth conic.* E. P. EVANS.

Solution by W. F. BEARD.

Let the centre locus meet the sides of ABC in L, M, N, and take on these sides L_1, M_1, N_1, harmonic conjugates of L, M, N.

Then LM_1N_1, L_1MN_1, L_1M_1N are straight lines, and the polars of L, M, N with respect to E are parallel lines through L_1, M_1, N_1: let them meet LMN in L_2, M_2, N_2, and let A' be the mid-point, and P the pole of BC, so that P is the intersection of OA' and L_1L_2. Then

$$P\{NA'ML_1\}=\{NOML_2\}=\{N_2\infty M_2L\}$$

(since LL_2, MM_2, NN_2 form involution, centre O)

$$=\frac{M_2L}{N_2L}=\frac{M_1L}{N_1L},$$

which is constant. Therefore the locus of P is a conic through A', L_1, M, N, and since AA', BB', CC', and AL_1, BM_1, CN_1 are concurrent triads, A', B', C', L_1, M_1, N_1 all lie on one conic.

Solution by R. F. DAVIS *and* C. V. DURELL.

Let $\Sigma l\beta\gamma=0$ be the trilinear equation of the conic circumscribing ABC. The coordinates of the centre are proportional to $l(-al+bm+cn)$, ..., and the centre always lying on a fixed straight line $\Sigma Pl^2-2\Sigma P'mn=0$. The coordinates of the pole of BC are $(-l, m, n)$, and it therefore lies on the conic $\Sigma P\alpha^2-2P'\beta\gamma+2Q'\gamma\alpha+2R'\alpha\beta=0$, which conic intersects BC in the same points as the conic $\Sigma P\alpha^2-2\Sigma P'\beta\gamma=0$.

Mr. Davis solves also by projecting B, C into the focoids.

417. [L¹. 5; 17. e.] *Eight normals to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touch any concentric co-axial ellipse $\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$. Prove this, and find the conditions for reality.* Trip., 1895.

Solution by W. F. BEARD, R. F. DAVIS *and* C. V. DURELL.

$\lambda x+\mu y=1$ is a normal to first ellipse if

$$a^2/\lambda^2+b^2/\mu^2=(a^2-b^2)^2,$$

and a tangent to the second if

$$a'^2\lambda^2+b'^2\mu^2=1.$$

Hence, eliminating λ, we have

$$b'^2(a^2-b^2)^2\mu^4+\mu^2\{a^2a'^2-b^2b'^2-(a^2-b^2)^2\}+b^2=0, \quad \text{.....A}$$

giving four values of μ, for each of which there are two values of λ, giving in all eight solutions.

If all the solutions are real, both values of μ^2 given by A must be real and positive. Hence

$$(a^2a'^2-b^2b'^2)^2-2(a^2-b^2)^2(a^2a'^2+b^2b'^2)+(a^2-b^2)^4>0,$$

i.e.

$$\{(a^2-b^2)^2-(aa'+bb')^2\}\{(a^2-b^2)^2-(aa'-bb')^2\}>0;$$

and also

$$(a^2-b^2)^2>(a^2a'^2-b^2b'^2);$$

and both these conditions are satisfied if

$$a^2-b^2>aa'+bb',$$

and this ensures also that the values of λ^2 are real and positive.

424. [R. 1. a.] *Two points moving at equal speeds, one on a circle, the other on a tangent, arrive at the point of contact simultaneously. Where does the line joining them meet the diameter through the point of contact* just before *they coincide?* C. E. M'VICKER.

Solution by J. BLAIKIE, J. F. HUDSON, A. W. POOLE, *and* PROPOSER.

Let O be the centre of the $\odot$, P, T two positions of the moving points and let TP meet the diameter through A, the point of contact in X, and let $POA=\theta$. Then

$$\frac{AX}{AT}=\frac{AX-NX}{AT-PN}=\frac{OA-ON}{AP-PN}=\frac{1-\cos\theta}{\theta-\sin\theta}=\frac{\frac{\theta^2}{\lfloor 2}-\frac{\theta^4}{\lfloor 4}+\dots}{\frac{\theta^3}{\lfloor 3}-\frac{\theta^5}{\lfloor 5}+\dots};$$

$$\therefore\ AX=AO\frac{3-\frac{\theta^2}{4}+\dots}{1-\frac{\theta^2}{20}+\dots}=3AO, \text{ when } \theta=0.$$

The result may be deduced from Snell's approximation, $\theta=\frac{3\sin\theta}{2+\cos\theta}$.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) **For Sale.**

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I-IX. Edited and Published by E. HENDRICKS, M.A., Des Moines, Iowa, U.S.A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor.* 1879-1881. Edited by ARTEMAS MARTIN, M.A. (Erie. Pa.)].

The Mathematical Monthly. Vols. I.-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. RUNKLE, A.M.

(2) **Wanted.**

Vols. I.-IV. *Mathésis.* 1881-1884.

The Messenger of Mathematics. Vols. 2, 15-20, 24, 25.

At the suggestion of Professor G. H. Bryan, F.R.S., this column is opened for the benefit of members of the Association. To members no charge will be made for insertion of requirements. Replies and enquiries should be addressed to the Editor when no postal address is attached to the items in the column.

BOOKS, ETC., RECEIVED.

Practical Exercises in Geometry. By W. D. EGGAR. Pp. xii., 287. 2s. 6d. 1903. (Macmillan.)

Elementary Arithmetic of the Octimal Notation. By G. H. COOPER. Pp. 70. n. p. 1902. (Whitaker Ray, San Francisco.)

Elements of Geometry. By R. LACHLAN and W. C. FLETCHER. Pp. xii., 208. 2s. 6d. 1903. (Arnold.)

Plane Geometry. By T. PETCH. Pp. v., 112. 2s. 1903. (Arnold.)

Intermediate Mathematics. University Tutorial Series. Sixth edition. 1902. (Clive.)

Récréations Arithmétiques. By E. FOURREY. 2nd. edition. Pp. 261. 8 fr. 1901. (Nony.)

Key to "The Elements of Hydrostatics." By S. L. LONEY, M.A. Pp. 146. 1902. (Cam. Univ. Press.)

A Short Introduction to Graphical Algebra. By H. S. HALL. 2nd edition. Pp. 49. 1903. 1s. (Macmillan.)

Euclid V., VI., XI. By RUPERT DEAKINS. Pp. 144. 1s. 6d. 1903. (Clive.)

The Elementary Geometry of Conics. By DR. C. TAYLOR, Master of St. John's College, Cambridge. Eighth edition revised, with a chapter on Inventio Orbium. Pp. viii., 156. 5s. 1903. (Deighton Bell.)

Über die Maxwell-Hertz'sche Theorie. By C. NEUMANN. No. II. Pp. 213-348. No. VIII. Pp. 755-860. From Vol. 27 der Abhandlungen der math.-physiken Classe der Königl. Sächsischen Gesellschaft der Wiss. 3 m. 50 pf. 1901, 1902. (Teubner.)

De L'Expérience en Géométrie. By C. DE FREYCINET. Pp. xix., 178. 4 fr. 1903. (Gauthier-Villars.)

Leçons Élémentaires sur la Théorie des Fonctions Analytiques. By E. A. FOUËT. Part I. Chaps. I.-V. Pp. 320. 7 fr. 50 c. 1902. (Gauthier-Villars.)

Traité de Mécanique Rationelle. By PAUL APPELL. Vol. III.

Équilibre et Mouvement des Milieux Continus. By PAUL APPELL. Vol. III. of *Traité de Mécanique Rationelle.* Pp. 560. 17 fr. 1903. (Gauthier-Villars.)

Geometrie der Dynamen. By E. STUDY. Part II. Pp. xiv., 241-603. 13 m. 40 pf. 1903. (Teubner.)

Wahrscheinlichkeitsrechnung. Part II. By E. CZUBER. Pp. 305-594. 12 m. 1903. (Teubner.)

Problèmes de Mécanique. By F. J. Pp. 574. 1903. (Ch. Poussielgue.)

An Elementary Treatise on the Mechanics of Machinery. By J. N. LE CONTE. Pp. x., 311. $2.25. 1902. (The Macmillan Company.)

The Thermodynamics of Heat Engines. By S. A. REEVE. Pp. xi., 316, 42. 1903. (The Macmillan Company.)

A Treatise on Differential Equations. By A. R. FORSYTH. Third Edition. Pp. xvi., 511. 14s. 1903. (Macmillan.)

Beginners' Algebra. By M. S. DAVID. Pp. vi., 232. 2s. 6d. 1903. (Black.)

Solid Geometry. By F. HOCEVAR. (Translated and adapted by C. Godfrey and E. A. Price.) Pp. viii., 80. 1s. 6d. 1903. (Black.)

A School Geometry. Parts I. and II. By H. S. HALL and F. H. STEVENS. Pp. x., 140. 1s. 6d. 1903. (Macmillan.)

Formulaire Mathématique. By G. PEANO. Pp. xvi., 313-407. 1903. (Bocca, Turin.)

Mathematics in the Elementary School. By D. E. SMITH, and F. M. M'MURRY. Teachers College Record. March, 1903. Pp. 70. 30 c. (Macmillan Co.)

The Monist. Jan.—April, 1903. 2$ per annum. (Kegan Paul.)

Open Court. Jan.—April, 1903. 1$ per annum. (Kegan Paul.)

ERRATA, ETC.

Note [123. 1². 1. a.] Mr. Davis is informed that the method given on p. 262 is to be found in Johnston's *Analytical Geometry* (Macmillan), p. 271.

p. 213, lines 18, 19, 20, 22, and in first note read w for ω.
line 23 read $\mu w + \omega\rho$ for $\mu\omega + \omega\rho$.
line 25 read $wq_2 + \omega(wq_1 + q\rho - q_2\rho) + \mu wq$
for $\omega q_2 + \omega(\omega q_1 + q\rho - q_2\rho) + \mu\omega q$.
p. 263, line 23 read *prime* for *primer*.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO.

V

I

A